Phys 410
Spring 2013
Lecture #43 Summary
8 May, 2013

We started in to relativistic dynamics with an effort to define relativistic momentum. We expect the laws of physics to have the same form in all reference frames, hence they should be Lorentz invariant. The easiest way to do this is to formulate the laws in terms of 4-vector quantities. They should also reduce to familiar Newtonian forms in the small-velocity limit.

Mass is defined to be an invariant quantity (all inertial reference frames agree on its value) and it is equal to the rest mass. Ordinary 3-momentum is not Lorentz invariant. 3-momentum that is conserved in a collision witnessed in one reference frame will not be conserved in another one moving by at relativistic speed (see problem 15.54). We need to develop a 4-vector version of momentum. Start with $x^{(4)} = (\vec{x}, ct)$, and consider taking a derivative with respect to time. The problem is that different inertial observers cannot agree on the evolution of time, hence we need a version of time that all observers can agree upon. This would be the proper time interval dt_0 which is the differential change in time when the particle of interest is at rest in your reference frame, corresponding to a differential 4-vector of $dx_0^{(4)} = (0, c\,dt_0)$. Comparing the invariant length of this 4-vector to that of a general differential displacement $dx^{(4)} = (\vec{v}, c)dt$ yields $dt_0 = dt/\gamma(v)$, where $\gamma(v) = 1/\sqrt{1-(v/c)^2}$ is the γ-factor associated with particle's velocity as measured in a given reference frame.

With this we can define a *bone-fide* velocity 4-vector that transforms like the space-time 4-vector: $u^{(4)} = \frac{dx^{(4)}}{dt_0} = \gamma(v)(\vec{v}, c)$. We define the associated Lorentz-invariant momentum as $p^{(4)} = m\gamma(v)(\vec{v}, c)$. Note that the 3-vector part of this reduces to the ordinary Newtonian momentum in the $\frac{v}{c} \ll 1$ limit, as required. Note that momentum now carries a fourth component – is this excess baggage or something useful? Recall Noether's theorem (and the idea of ignorable coordinates in Lagrangians), which says that the homogeneity of space implies linear momentum conservation. Likewise the homogeneity of time implies conservation of energy. In this case the time-like component of the momentum 4-vector is defined as relativistic energy E divided by the speed of light, $p^{(4)} = (m\gamma(v)\vec{v}, E/c)$. In other words $E = \gamma(v)mc^2$.

To examine the plausibility of this assignment of the relativistic energy, look at its value in the small-velocity limit $\frac{v}{c} \ll 1$. In this limit $E = \frac{mc^2}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \cong mc^2 + \frac{1}{2}mv^2 + \cdots$, through a binomial expansion of the denominator. The first term is called the rest energy and is a constant as far as classical Lagrangian mechanics is concerned, hence it plays no role in Newtonian dynamics. The second term is the Newtonian kinetic energy that we have employed since the get-go. Thus this definition of energy reduces to our familiar one in the low-speed Newtonian limit. The relativistic kinetic energy can be written as $T = E - mc^2 = (\gamma(v) - 1)mc^2$.

Finally, from the invariant length of the momentum-energy 4-vector we derived the famous relativistic connection between energy, momentum and the rest energy: $E^2 = (\vec{p}c)^2 + (mc^2)^2$, where $\vec{p}$ is the 3-vector part of the 4-vector $p^{(4)}$.